

# Brane world cosmological constant in the models with large extra dimensions.

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## Abstract

We consider “brane universe” with nonzero tension in the models with large extra dimensions. We find exact solutions of higher-dimensional Einstein equations with single flat Minkowsky brane of arbitrary large tension (or brane cosmological constant) and compact extra dimensions. The brane curves the bulk space-time in a small region around it. There is no fine tuning of energy scales in our model.

Phenomenological models with large extra space-time dimensions [1] have attracted considerable attention recently. Like in conventional Kaluza-Klein models, in the models with large extra dimensions higher-dimensional space-time is supposed to be a direct product  $\mathcal{M}^{4+n} = R^4 \times K^n$  of a four-dimensional plane with a compact manifold  $K^n$  (for example, an  $n$ -dimensional sphere  $S^n$  of radius  $R$ ). The metric on  $\mathcal{M}^{4+n}$  is just a sum of flat Minkowsky metric on  $R^4 = \{x^\mu\}, \mu = 0, 1, 2, 3$  and a standard metric on  $K^n = \{y^a\}, a = 1, \dots, n$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{ab}(y) dy^a dy^b \quad (1)$$

The key difference of models with large, or infinite extra dimensions from the conventional Kaluza-Klein models is that matter fields of the Standard Model

of particle physics are confined to a four-dimensional surface (3-brane)  $M^4$  embedded into  $\mathcal{M}^{4+n}$ . The brane  $M^4$  can be viewed as a topological defect in higher-dimensional space. The Standard Model particles can be localized on this defect either due to some field-theoretical mechanism [2, 3] or in a way supposed by String Theory [4].

If matter fields of the Standard Model are localized on the brane, the size  $R$  of extra dimensions is not limited to be very small. If  $R$  is large enough, the higher-dimensional Planck mass  $M_{pl(4+n)}$  which is related to the four-dimensional one  $M_{pl(4)} \sim 10^{16}$  TeV as

$$M_{pl(4+n)}^{n+2} = M_{pl(4)}^2 R^{-n} \quad (2)$$

can be as small as several TeV, which opens up a possibility of testing the presence of extra dimensions and quantum gravity effects in future collider experiments.

Even if the higher-dimensional Planck mass is much larger than TeV, phenomenological models with brane universe have generically another testable feature [5, 6]. The presence of the brane brakes spontaneously a part of translational invariance in higher-dimensional space-time. Goldstone bosons associated to this symmetry breaking behave themselves in first approximation as massless scalar particles localized on the brane. These particles interact with Standard Model fields. Such interaction produces an essential contribution to particle collisions on energy scales higher or approximately equal to  $\mu = \Lambda^{1/4}$  where  $\Lambda$  is the typical energy density on the brane. Even if  $M_{pl(4+n)}$  is essentially higher than 1 TeV, the energy scale  $\mu$  can be accessible to accelerators. The lower bound on  $\mu$  imposed by requirement of consistency with electroweak physics [7] and astrophysical bounds [8] turns out to be

$$\mu \geq 10^{-1} \text{ TeV} \quad (3)$$

The scale of the energy density on the brane  $\Lambda$  must, therefore, be much larger than the natural cosmological energy density scale

$$\Lambda = \mu^4 \gg \rho_{cr} = \frac{3H^2}{8\pi G_4} \sim 10^{-58} \text{ TeV}^4 \quad (4)$$

where  $\rho_{cr}$  is the critical energy density of the universe,  $H$  is the Hubble constant and  $G_4$  is the four-dimensional gravitational constant. All the energy density  $\rho \sim \rho_{cr}$  provided by the visible matter in the universe is just a small

perturbation of the energy density  $\Lambda$  of the brane. The energy density  $\Lambda$  can be ascribed, for example to the topological defect itself or to the vacuum energy of the Standard Model fields localized on the brane.

The higher dimensional space-time  $\mathcal{M}^{4+n}$  must be a solution of higher-dimensional Einstein equations with compact extra dimensions and a brane with flat Minkowsky geometry and nonzero tension  $\Lambda$ . The general maximally symmetric stress-energy tensor of the brane has the form

$$T_{\mu\nu} \sim -\Lambda \gamma_{\mu\nu} \delta(M^4) \quad (5)$$

where  $\gamma_{\mu\nu}$  is the induced metric on the brane which is equal to  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  for the Minkowsky brane.

Since the stress-energy tensor of the brane is not zero, the brane must curve bulk space-time around it. In general, it is not obvious that there exists a *static* solution of higher-dimensional Einstein equations with *single* Minkowsky brane of nonzero tension  $\Lambda$  and *compact* extra dimensions. Moreover, we know that in the absence of extra dimensions the presence of nonzero stress-energy tensor of the form (5) leads to exponential expansion of the universe. In the presence of extra dimensions large vacuum energy can curve extra dimensions, leaving the four “physical” dimensions flat [9]. There exists a number of static solutions of higher-dimensional Einstein equations with bulk cosmological constant and a brane of nonzero tension [10]-[17]. But in the known static solutions the extra dimensions are either infinite [12, 13, 14] or they are finite but there is another “hidden” brane [11, 17] or curvature singularity of another kind [15, 16] at a finite distance from the “observable” four-dimensional brane. It is possible to get rid of the “hidden” brane making the compact extra dimensions nonstatic [10].

In what follows we present solutions of higher-dimensional Einstein equations with a flat brane of nonzero tension  $\Lambda$  and compact extra dimensions. The scale of  $\Lambda$  is not fine tuned neither to the higher-dimensional Planck scale  $M_{pl(4+n)}^4$  nor to the Kaluza-Klein scale  $R^{-4}$  and is defined by internal properties of the brane.

Typical size of the space-time region around the brane in which the background geometry is essentially deformed by the presence of the brane is much smaller than both the higher-dimensional Planck length and the size of the extra dimensions even if the brane tension is as high as  $M_{pl(4+n)}^4$ . Therefore, in description of classical and quantum fluctuations of the brane around its

equilibrium position we can use the approximation of test brane on a fixed background space-time in a way supposed in [5, 6, 7, 8].

We consider a space-time  $\mathcal{M}^{4+n} = R^4 \times S^n$  with compact extra dimensions with topology of the sphere  $S^n$ . Let us restrict our attention to the case  $n = 4$ , although the construction is easily generalized for arbitrary  $n \geq 4$ . The set of coordinates on  $M^4 \times S^4$  is

$$x^A = (\underbrace{x^\mu}_{R^4}, \underbrace{r, \psi, \theta, \phi}_{S^4}) \quad (6)$$

In the absence of the brane the metric on  $S^4$  is the standard metric on a sphere of radius  $R$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 - \frac{r^2}{R^2}\right) d\psi^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (7)$$

The coordinate  $r$  is defined within the interval  $r \in (0, R)$ , the coordinate  $\psi$  is periodic with the period  $2\pi R$  and  $(\theta, \phi)$  are the standard angular coordinates on a two-dimensional sphere. The coordinate system  $(r, \psi, \theta, \phi)$  becomes singular at  $r = 0, R$ . In the vicinity of  $r = 0$  the radius of two-dimensional sphere parameterized by  $(\theta, \phi)$  goes to zero so that the region  $0 \leq r < \hat{r}$  for some small  $\hat{r}$  has topology  $D^3 \times S^1$  of direct product of three-dimensional disk parameterized by  $(r, \theta, \phi)$  and a circle parameterized by  $\psi$ . Singularity of coordinate system at  $r = 0$  is just the singularity of three-dimensional spherical coordinates at  $r = 0$ . In the vicinity of  $r = R$  radius of the circle  $S^1$  parameterized by  $\psi$  goes to zero and topology of a region  $\hat{r} < r \leq R$  is  $S^2 \times D^2$  where  $S^2$  is parameterized by  $(\theta, \phi)$  and two-dimensional disk  $D^2$  is parameterized by  $(r, \psi)$ . Singularity of the coordinate system at  $r = R$  is the same as at the origin of polar coordinates on a two-dimensional plane. The whole sphere  $S^4$  is a connected sum  $S^4 = (D^3 \times S^1) \# (S^2 \times D^2)$  of the above two manifolds with boundaries.

The space-time (7) is a solution of the eight-dimensional Einstein equations with stress-energy tensor

$$\begin{aligned} T^\mu_\nu &= -\rho \delta^\mu_\nu \\ T^a_b &= -w \rho \delta^a_b \end{aligned} \quad (8)$$

where the energy density  $\rho$  and parameter  $w$  which describes the equation of

state are

$$\begin{aligned}\rho &= \frac{3}{4\pi G_8 R^2} \\ w &= 1/2\end{aligned}\tag{9}$$

Here  $G_8$  is the eight-dimensional gravitational constant.

Let us now consider a brane  $M^4$  embedded into  $\mathcal{M}^8$  as  $M^4 : \{r < \hat{r}\}$  where  $\hat{r} \ll R$  is a small brane thickness. Presence of the brane destroys the maximally symmetric form of the bulk metric (7). A general form of metric in the bulk is now

$$ds^2 = e^{A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{B(r)} \psi^2 + e^{C(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2) \tag{10}$$

with arbitrary functions  $A(r), B(r), C(r)$ . Outside the brane the metric must be a solution of eight-dimensional Einstein equations with the same bulk stress-energy tensor (8). The most simple choice is

$$ds_{r>\hat{r}}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 - \frac{r_0}{r} - \frac{r^2}{R^2}\right) d\psi^2 + \left(1 - \frac{r_0}{r} - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{11}$$

the sum of Minkowsky metric along  $x^\mu$  directions and Euclidian Schwarzschild-deSitter black hole with gravitational radius  $r_0$ . The coordinate  $\psi$  is defined now with the period  $2\pi\tilde{R}$  where  $\tilde{R}$  is chosen in such a way that the space-time has no conical singularity at  $r = r_1$  where  $r_1$  is the larger root of cubic equation  $r^3 + R^2(r_0 - r) = 0$ . If we are interested in the case when both the thickness  $\hat{r}$  and the gravitational radius  $r_0$  are much smaller than  $R$ , then  $r_1 \approx \tilde{R} \approx R$ .

The metric “inside” the brane in the region  $r < \hat{r}$  depends on details of the brane structure and behavior of the functions  $A(r), B(r), C(r)$  (10) in the vicinity of  $r = 0$  can be different for different types of four-dimensional topological defects in higher-dimensional space. The metric in the region  $0 < r < \hat{r}$  inside the brane must satisfy the boundary condition

$$ds_{r=\hat{r}}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 - \frac{r_0}{\hat{r}} - \frac{\hat{r}^2}{R^2}\right) d\psi^2 + \hat{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{12}$$

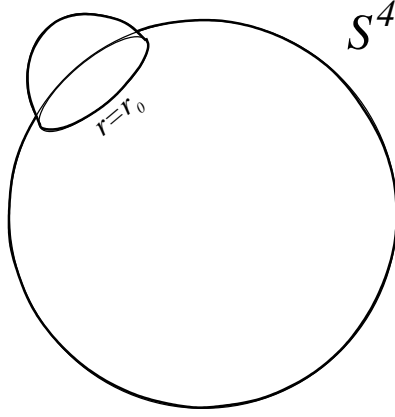


Figure 1: The geometry of the sphere  $S^4$  deformed by the presence of the brane inside  $r = \hat{r}$ .

in order to match the outer metric (11) continuously. For example, let us take the metric

$$ds_{r < \hat{r}}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 - \frac{r^2}{a}\right) d\tilde{\psi}^2 + \left(1 - \frac{r^2}{a}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (13)$$

In this case the complete space-time with the brane is constructed as follows. We cut a small region  $0 < r < \hat{r}$  of the four-dimensional sphere  $S^4$  of large radius  $R$  and glue a space locally isometric to a smaller sphere of radius  $\sqrt{a}$  on inside  $r = \hat{r}$  as it is schematically presented on Fig. 1.

The coordinate  $\tilde{\psi}$  is defined with an arbitrary period  $\tilde{\psi} \in (0, 2\pi L)$  and the relation between  $a$  and  $L$  is found from the requirement that the induced metric on the surface  $r = \hat{r}$  is equal to (12)

$$L^2 \left(1 - \frac{\hat{r}^2}{a}\right) = R^2 \left(1 - \frac{r_0}{\hat{r}} - \frac{\hat{r}^2}{R^2}\right) \quad (14)$$

The matter stress-energy tensor in the region  $0 < r < \hat{r}$  can be calculated from the Einstein equations

$$T_B^A = \frac{1}{8\pi G_8} \left( R_B^A - \frac{1}{2} \delta_B^A R \right) \quad (15)$$

It has the form (8) with

$$\begin{aligned}\rho_{r<\hat{r}} &= \frac{3}{4\pi G_8 a} \\ w_{r<\hat{r}} &= 1/2\end{aligned}\tag{16}$$

The surface stress-energy tensor at  $r = \hat{r}$  can be found from Israel junction conditions across  $r = \hat{r}$  which relate it to the jump of extrinsic curvature  $K_B^A$  of the surface  $r = \hat{r}$

$$T_B^A = \frac{1}{8\pi G_8} ([K_B^A] - \delta_B^A [K])\tag{17}$$

(square brackets denote the jump  $[K] = K(\hat{r} + 0) - K(\hat{r} - 0)$ ). Substituting the metrics (13) and (11) into (17) we find

$$T_{\nu(r=\hat{r})}^\mu = \frac{3r_0 R^2(2R - L) + 2\hat{r}R(L^2 + 2RL - 3R^2) + 6\hat{r}^3(R - L)}{16\pi G_8 \hat{r}^2 L^2 R \sqrt{1 - \hat{r}^2/a}} \delta_\nu^\mu\tag{18}$$

The effective four-dimensional stress-energy tensor of brane is obtained by integrating the eight-dimensional stress-energy tensor over the brane volume

$$\mathcal{T}_\nu^\mu = 4\pi \int_0^{\hat{r}} r^2 dr \int_0^{2\pi L} d\psi T_{\nu(r<\hat{r})}^\mu + 4\pi \hat{r}^2 \int_0^{2\pi L} d\psi \sqrt{1 - \hat{r}^2/a} T_{\nu(r=\hat{r})}^\mu\tag{19}$$

Substituting (16) and (18) into (19) we get

$$\mathcal{T}_\nu^\mu = -\Lambda \delta_\nu^\mu\tag{20}$$

where the brane tension, or the cosmological constant on the brane is

$$\Lambda = \frac{\pi}{G_8} \left( \left( \frac{3}{R} - \frac{1}{L} \right) \hat{r}^3 + \left( L - 2R + \frac{R^2}{L} \right) \hat{r} + \left( \frac{3}{2}R - L \right) r_0 \right)\tag{21}$$

Depending on the values of parameters  $L, \hat{r}, r_0$  the brane tension can be both positive or negative. We consider several particular cases. If take  $L \rightarrow 0$  so that the size of  $S^1$  parameterized by  $\psi$  is very small at  $r = 0$ , the brane tension is approximately

$$\Lambda \approx \frac{\pi \hat{r} R^2}{G_8 L}\tag{22}$$

If we choose  $L \approx R$  so that the size of  $S^1$  is large at  $r = 0$ , but becomes small at  $r = \hat{r}$  if  $\hat{r} \approx r_0$ , the brane tension is

$$\Lambda \approx \frac{\pi r_0 R}{2G_8} \quad (23)$$

Thus, the cosmological constant on the brane depends on the internal structure of the brane.

Let us discuss the question of fine tuning of parameters in our model. Within four-dimensional General Relativity cosmological constant must be fine tuned to zero in order to explain flat Minkowsky geometry of space-time. In the models with extra dimensions the presence of nonzero cosmological constant on the brane (5) does not necessary cause the exponential expansion of the brane universe. The brane tension (21) is not fine tuned neither to Kaluza-Klein scale  $R^{-4}$  nor to the Planck scale  $M_{pl(8)}^4$ . Flat Minkowsky geometry of the brane universe is explained in our model by a special choice of matter equation of state, rather than fine-tuning of the energy scale.

Indeed, let us first consider the space-time  $\mathcal{M}^8 = R^4 \times S^4$  without brane. If the geometry of  $R^4$  sections of this space-time is Minkowsky,  $\mathcal{M}^8$  is a solution of eight-dimensional Einstein equations with stress-energy tensor (8), (9). Such a stress-energy tensor can be generated, for example by a massless scalar field  $\Psi^a : \mathcal{M}^8 \mapsto S^4$  which takes values on a four-dimensional sphere. The stress-energy tensor of this field is

$$T_{AB} = \Psi_{,A}^a \Psi_{,B}^a - \frac{1}{2} g_{AB} g^{CD} \Psi_{,C}^a \Psi_{,D}^a \quad (24)$$

If we consider a topologically nontrivial configuration where  $\Psi$  “winds” once on the sphere

$$\Psi^a(x, y) = \text{const} \cdot y^a \quad (25)$$

the parameter  $w$  (9) which describes the equation of state for  $\Psi$  is

$$w_\Psi = 1/2 \quad (26)$$

Now, suppose that the geometry of the  $R^4$  part of  $\mathcal{M}^8$  is deSitter or anti-deSitter with cosmological constant  $\Lambda$ , rather than Minkowsky. Such a space-time is a solution of eight-dimensional Einstein equations with the stress-energy tensor of the form (8) with energy density

$$\rho = \frac{1}{8\pi G_8} \left( \Lambda + \frac{6}{R^2} \right) \quad (27)$$

and equation of state

$$w = \frac{3 + 2\Lambda R^2}{6 + \Lambda R^2} \quad (28)$$

We see that if  $\Lambda \neq 0$  the matter equation of state must be different from (9). Thus, if we take matter with equation of state  $w = 1/2$ , for example, the scalar field  $\Psi$  in topologically nontrivial vacuum (25), the four-dimensional sections of  $\mathcal{M}^8 = R^4 \times S^4$  will have flat geometry.

In the same way geometry of the brane is flat if the equation of state of matter in the region  $r \leq \hat{r}$  inside and on the boundary of the brane is adjusted in appropriate way.

The size of space-time region around the brane in which the background metric is essentially deformed by the brane is very small  $r_0 \ll M_{pl(8)}^{-1} \ll R$  even for a brane of very high tension. This enables us to treat the brane as a test brane on a fixed background. The situation is analogous to the one encountered in particle physics where one can neglect gravitational (self) interaction of particles because their gravitational radius is much smaller than the Compton wavelength. Fluctuations of the brane around its equilibrium position can be excited in particle collisions or astrophysical processes at the energy scales close to  $\mu = \Lambda^{1/4}$  (3) [5, 6, 7, 8].

We note also that cosmological models with a brane universe of nonzero tension must be very different from the conventional four-dimensional ones. Indeed, the large four-dimensional cosmological constant  $\Lambda$  does not cause exponential expansion of the universe in our model. If we add some matter on the brane with equation of state  $p = -\rho$  ( $p$  and  $\rho$  are four-dimensional pressure and energy density), the brane universe still can remain static, since we just change the brane cosmological constant  $\Lambda \rightarrow \Lambda + \rho$  which would lead to the change of gravitational radius  $r_0$  of the brane. The expansion of the brane universe is caused only by matter with equation of state different from  $p = -\rho$ .

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